Eur. Phys. J. C 21, 145–153 (2001)
Digital Object Identifier (DOI) 10.1007/s100520100717

THE EUROPEAN
PHYSICAL JOURNAL C

Quark–hadron duality, factorization and strong phases in $B_d^0 \to \pi^+\pi^-$ decay

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Received: 19 March 2001 / Revised version: 25 May 2001 /
Published online: 29 June 2001 –

Abstract. We consider the hadronic description of the $B_d^0 \to \pi^+\pi^-$ decay, with the aim to investigate the strong phases generated by the final state interactions. The derivation of the dispersion relations using the Lehmann–Symanzik–Zimmermann formalism and the Goldberger–Treiman method to include inelastic effects in the spectral function are presented. We discuss the problem of quark–hadron duality and estimate in the hadronic formalism the corrections to the factorized amplitude in the heavy quark limit.

1 Introduction

The inclusion of the strong interaction effects in the theory of exclusive nonleptonic B decays is a very difficult task. The problem has been investigated recently by many authors, in particular, for charmless decays into light pseudoscalar mesons, since the strong phases of these amplitudes are crucial for the determination of CP-violating phases in present and future experiments [1]. The first measurements of the branching fractions of the B decays into $\pi\pi$ and πK final states [2–4] considerably stimulated the theoretical and phenomenological work devoted to these processes in various approaches. In the so-called "naive factorization approximation" [5], the matrix elements of the operators entering the weak effective hamiltonian are expressed as products of meson decay constants and hadronic form factors, which are evaluated in a phenomenological way. An obvious deficiency of this approximation is the renormalization scale dependence of the results, expressed as μ-dependent Wilson coefficients multiplied by μ-independent hadronic form factors. Improvements to the factorization approximation were discussed in several papers [6–9]. Recent calculations of the $B \to \pi\pi$ decay amplitude were performed either in the generalized QCD factorization approach [9–14], or by more conventional perturbative QCD methods [15,16].

The nonleptonic B decays were also investigated recently in a hadronic approach, in which a part of the strong dynamics accompanying the weak decay is described using the unitarity of the S-matrix, dispersion relations and Regge phenomenology [17–27]. In the present paper, we apply this approach to the particular case of $B_d^0 \to \pi^+\pi^-$ decay. One aim of our study is to compare the predictions of the hadronic and the partonic treatments and to test the validity of quark–hadron duality.

In the next section, we discuss the derivation of dispersion relations with respect to the momentum squared of the external particles, by applying the Lehmann–Symanzik–Zimmermann (LSZ) reduction formalism [28] to the S-matrix element of the decay process. In Sect. 3 we explain the Goldberger–Treiman procedure to include inelastic contributions in the spectral function and apply it to the amplitudes of the decay $B_d^0 \to \pi^+\pi^-$. In Sect. 4 we consider the problem of quark–hadron duality, and estimate in the hadronic formalism the corrections to the factorized amplitude produced by the final state interactions in the heavy quark limit. Section 5 contains our conclusions.

2 Dispersion relations for the decay amplitude

We consider the decay amplitude

$$A(B_d^0 \to \pi^+\pi^-) = \langle \pi^+(k_1)\pi^-(k_2), \mathrm{out}|\mathcal{H}_\mathrm{w}(0)|B_d^0(p), \mathrm{in}\rangle, \quad (1)$$

where the "in" and "out" states are defined with respect to the strong interactions and $\mathcal{H}_\mathrm{w}$ is the weak effective hamiltonian density

$$\mathcal{H}_\mathrm{w} = \frac{G_\mathrm{F}}{\sqrt{2}} \sum_{j=u,c} V_{jd} V_{jb}^* \times \left[C_1(\mu) O_1^j(\mu) + C_2(\mu) O_2^j(\mu) + \sum_{i=3,\ldots,8} C_i(\mu) O_i(\mu) \right]. \quad (2)$$

In this relation, O_i are local $\Delta B = 1, \Delta S = 0$ operators, and C_i the corresponding Wilson coefficients, which take

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into account perturbatively the strong dynamics at distances shorter than $1/\mu$. Using the expression (2) of the weak hamiltonian, the decay amplitude (1) can be split in two terms:

$$A(B_d^0 \to \pi^+\pi^-) = V_{ud}V_{ub}^* A_u + V_{cd}V_{cb}^* A_c, \quad (3)$$

with the CP-violating phase $\gamma = \mathrm{Arg}(V_{ub}^*)$ appearing in the first term.

The physical amplitude (1) is calculated for $p = k_1 + k_2$ at on-shell values of the momenta, $p^2 = m_B^2$, $k_1^2 = m_\pi^2$, $k_2^2 = m_\pi^2$. The extrapolation to off-shell external momenta can be achieved by the LSZ reduction formalism [28]. In [25] we applied this technique to the expression (1) of the amplitude. As we shall see below, it is more convenient to start from the S-matrix element

$$S_{B_d^0\to\pi^+\pi^-} = \langle \pi^+(k_1)\pi^-(k_2), \mathrm{out}|B_d^0(p), \mathrm{in}\rangle, \quad (4)$$

where the transition from the "in" to the "out" states includes both the strong and weak interactions. The expression (1) of the decay amplitude is obtained by expanding the S-matrix to first order in the weak hamiltonian. However, we can apply the LSZ reduction to the B meson in (4), which leads to the alternative expression

$$A(B_d^0 \to \pi^+\pi^-) = \frac{1}{\sqrt{2p_0}}\langle \pi^+(k_1)\pi^-(k_2), \mathrm{out}|\eta_{B^0}(0)|0\rangle, \quad (5)$$

where $\eta_{B^0}(x) = \mathcal{K}_x\phi_{B^0}(x)$ is the source of the meson B_d^0 and ϕ_{B^0} its interpolating field ($\mathcal{K}_x$ is the Klein–Gordon operator). We recall that in a Lagrangian theory the source, which includes both the strong and weak interactions, can be written formally as

$$\eta_{B^0}(x) = \frac{\delta\mathcal{L}_{\mathrm{int}}}{\delta\phi_{B^0}} - \partial_\mu\frac{\delta\mathcal{L}_{\mathrm{int}}}{\delta\partial_\mu\phi_{B^0}}. \quad (6)$$

The matrix element in (5) can be defined for arbitrary $s = p^2 = (k_1+k_2)^2$, the physical amplitude corresponding to $s = m_B^2$. We notice that (5) is similar to the definition of the pion electromagnetic form factor, where η_B is replaced by the electromagnetic current J_μ. We can apply therefore the standard methods used in deriving the dispersion relations for the pion form factor [29–31]. More precisely, by the LSZ reduction of one final meson (say, π^+) in (5), we obtain

$$\begin{aligned} &A(B_d^0 \to \pi^+\pi^-)\\ &= \frac{\mathrm{i}}{\sqrt{4k_{10}p_0}}\int \mathrm{d}x \mathrm{e}^{\mathrm{i}k_1x}\theta(x_0)\langle\pi^-(k_2)|[\eta_{\pi^+}(x), \eta_{B^0}(0)]|0\rangle\\ &-\frac{\mathrm{i}}{\sqrt{4k_{10}p_0}}\int \mathrm{d}x\mathrm{e}^{\mathrm{i}k_1x}\delta(x_0)\\ &\times\langle\pi^-(k_2)|\mathrm{i}k_{10}[\phi_{\pi^+}(x), \eta_{B^0}(0)]\\ &-[\partial_0\phi_{\pi^+}(x), \eta_{B^0}(0)]|0\rangle, \end{aligned} \quad (7)$$

where $\eta_{\pi^+}(x)$ is the source of the reduced pion.

The second integral in (7) contains equal-time commutators produced by the action of the Klein–Gordon operator $\mathcal{K}_x$ upon the function $\theta(x_0)$. As shown in [30], the most general form of this term, called "degenerate", is a constant or a polynomial of the Lorentz invariant variables. To calculate the degenerate term, one needs the expression of the source η_B in terms of the interpolating field ϕ_{π^+} or its time derivative, which in a hadronic Lagrangian theory might be obtained from the formal expression (6). The commutators can then be evaluated in principle by applying the canonical commutation rules, satisfied, up to a normalization constant, by the interpolation fields [30]. However, in the standard model, the hadronic fields are defined in terms of the underlying quark and gluon degrees of freedom, and the definition of the off-shell fields might introduce ambiguities in the evaluation of the degenerate term (the result depends also on which pion, π^+ or π^-, is reduced, since their quark content is different).

We now turn to the first term of (7), which is usually called "dispersive term", and has a more complicated structure as a function of the squared external momenta. The integral defines a function holomorphic at the values of the momenta for which it is convergent. First, due to the presence of the $\theta(x_0)$ function, the integral upon x_0 in (7) is convergent for $\mathrm{Im}k_{10} > 0$, i.e. in the upper half of the complex k_{10} plane. A detailed analysis must exploit also the causality properties of the commutator [30], which restricts the integral over the spatial variables to $|\boldsymbol{x}| < |x_0|$. The difficult part of the conventional proofs of the dispersion relations is to show that the integrals over x_0 and $\boldsymbol{x}$ are convergent for complex values of the external momenta. As discussed in [30], it is sometimes useful to go to a particular Lorentz frame and consider a particular variable, for instance k_{10}, instead of trying to think in terms of Lorentz invariants. Also, it is useful to treat simultaneously the matrix elements $\langle\pi\pi, \mathrm{out}|\eta_B|0\rangle$, $\langle\pi\pi, \mathrm{in}|\eta_B|0\rangle$ and $\langle\pi|\eta_B|\pi\rangle$, which are represented by the same analytic function in various parts of the complex plane of the dispersive variable.

In the present case, it is convenient to choose the system with the unreduced pion π^- at rest ($\boldsymbol{k}_2 = 0$), when $k_{10} = (s - k_1^2 - m_\pi^2)/2m_\pi$. In what follows we shall either work with s variable keeping $k_1^2 = m_\pi^2$ fixed, or with k_1^2 variable at fixed $s = m_B^2$. By expressing $\boldsymbol{k}_1^2$ in terms of k_{10} and the fixed Lorenz invariant momentum, the first term of (7) depends only on the variable k_{10}, and is analytic in the complex k_{10} plane, except a possible discontinuity along the real axis, given by [30]

$$\begin{aligned} \sigma(k_{10}) = \frac{1}{2\sqrt{4k_{01}p_0}}\Bigg[&\sum_n \delta(k_1 + k_2 - p_n)\\ &\times\langle\pi^-(k_2)|\eta_{\pi^+}(0)|n\rangle\langle n|\eta_{B^0}(0)|0\rangle\\ &-\sum_n \delta(k_1 + p_n)\langle\pi^-(k_2)|\eta_{B^0}(0)|n\rangle\\ &\times\langle n|\eta_{\pi^+}(0)|0\rangle\Bigg]. \end{aligned} \quad (8)$$

This expression is obtained formally from (7) by replacing $\mathrm{i}\theta(x_0)$ by $1/2$, inserting a complete set of intermediate states in the commutator and using translational invariance [30]. In order to evaluate the spectral function, we recall that the sources contain both the strong and the weak interactions, the last ones being treated to first order. As we shall see below, the spectral function takes different forms, depending on the external momentum adopted as dispersive variable.

Let us assume first that k_1^2 is fixed at the physical value $k_1^2 = m_\pi^2$, and treat the amplitude as a function of the variable $s = (k_1 + k_2)^2$. In the reference system chosen above $k_{10} = (s - 2m_\pi^2)/2m_\pi$ and $\boldsymbol{k}_1^2 = k_{10}^2 - m_\pi^2$. It is easy to see that in this case the second sum in the expression (8) has no contribution. Indeed, since $k_1^2 = m_\pi^2$, the only state which contributes is the one-pion state $|n\rangle = |\pi\rangle$, and $\langle\pi|\eta_\pi|0\rangle = 0$ [30].

As concerns the first sum of (8), the intermediate states n which contribute are of two kinds: the first ones are generated by the weak part of the source η_B in the second matrix element, and undergo a strong transition to the final state $\pi^+\pi^-$, mediated by the strong part of η_π. According to (5), the second matrix element is equal to the weak decay amplitude of an off-shell meson B, with momentum squared equal to s, while the first matrix element is a strong amplitude, evaluated at the c.m. momentum squared equal to s. Therefore, the contribution of these states in the unitarity sum represents the so-called "final state interactions" (FSI). The lowest intermediate state consists of two pions, which produces the lowest branch point at $k_{10} = m_\pi$, or, equivalently, $s = 4m_\pi^2$.

The intermediate states n of the second type are produced by the strong part of the source η_B in the second matrix element, which describes the strong decay of an off-shell B meson. The first matrix element, where the contribution is of the weak part of η_π, describes the weak transition amplitudes from the state n to the final $\pi^+\pi^-$ state. These terms are usually interpreted as "initial state interactions" (ISI). The lowest physical state which can contribute is the pair $B^*\pi$, producing the lowest threshold $s = (m_{B^*} + m_\pi)^2$.

The whole amplitude can be recovered from the discontinuity by means of a dispersion integral. To write it down, we need the asymptotic behavior of the discontinuity, which is difficult to estimate, since it involves off-shell quantities. Assuming, for simplicity, that one subtraction is necessary, and combining the possible degenerate terms discussed above with the subtraction constant of the dispersive part, we express the physical amplitude as

$$\begin{aligned} &A(B^0 \to \pi^+\pi^-) \\ &= A(s_0) + \frac{m_B^2 - s_0}{\pi} \int_{4m_\pi^2}^{\infty} \mathrm{d}s \frac{\sigma_{\mathrm{FSI}}(s)}{(s - m_B^2 - \mathrm{i}\epsilon)(s - s_0)} \\ &\quad + \frac{m_B^2 - s_0}{\pi} \int_{(m_B^* + m_\pi)^2}^{\infty} \mathrm{d}s \frac{\sigma_{\mathrm{ISI}}(s)}{(s - m_B^2)(s - s_0)}, \end{aligned} \tag{9}$$

where

$$\sigma_{\mathrm{FSI}} \approx \sum_n \delta(k_1 + k_2 - p_n) M^*(n \to \pi\pi) A(B \to n),$$

$$\sigma_{\mathrm{ISI}} \approx \sum_n \delta(k_1 + k_2 - p_n) A^*(n \to \pi\pi) M(B \to n) \tag{10}$$

are the spectral functions associated to the final (initial) state interactions, respectively. In these relations, $A(M)$ denote the amplitudes of the weak (strong) transitions, respectively, evaluated for an off-shell B momentum squared equal to s. Since B is stable with respect to the strong interactions, $m_B < m_{B^*} + m_\pi$, the initial state interactions (the last integral in (9)) do not contribute to the on-shell imaginary part. We mention that a dispersion relation similar to (9) was derived recently in [32] for $K \to \pi\pi$ decay, starting from the definition (1) of the decay amplitude, treating the weak hamiltonian $\mathcal{H}_{\mathrm{w}}$ as the source of a spurion, and using the Mandelstam representation.

As mentioned above, the expression (7) can be analytically continued also in the variable k_1^2, at fixed s, equal to the physical value $s = m_B^2$. In this case, in the reference system chosen above, $k_{10} = (m_B^2 - k_1^2 - m_\pi^2)/2m_\pi$ and $\boldsymbol{k}_1^2 = [k_1^2 - (m_B + m_\pi)^2][k_1^2 - (m_B - m_\pi)^2]/(2m_\pi)^2$. The spectral function is given formally by the same expression (8), but now the contributions are different. First, we notice that the second sum in (8), which previously vanished on account of $k_1^2 = m_\pi^2$, now includes terms which are produced by the strong and the weak parts of the source η_π, for a variable k_1^2. Just like in the discussion above, there are intermediate states which are generated by the strong decay of an off-shell pion, and then undergo a weak transition mediated by the weak part of η_B, and also intermediate states which are produced by the weak part of η_π and generate afterwards a pion and a B meson through a strong interaction. In the first case the lowest branch point is at $k_1^2 = 9m_\pi^2$, corresponding to the intermediate state with three pions, and in the second case at $k_1^2 = (m_B + m_\pi)^2$, corresponding to the intermediate state πB. From the connection between k_{10} and k_1^2 in the particular system mentioned above, one can see that the branch cut corresponds to negative values of k_{10}. This means that these contributions originate actually from the matrix element $\langle\pi|\eta_B|\pi\rangle$, related to the $B \to \pi\pi$ decay amplitude by crossing symmetry.

As concerns the first sum in the spectral function (8), the intermediate states which bring about a nonvanishing contribution have $p_n^2 = (k_1 + k_2)^2$ fixed at the value m_B^2. The strong part of η_B therefore gives no contribution, since the lowest state possible (the pair $B^*\pi$) cannot be produced at this energy. On the other hand, the weak part of the source η_B has a nonvanishing contribution, producing intermediate particles which then undergo a strong interaction. This contribution represents therefore the final state interactions. The delta function implies the condition $k_1^2 = (p_n - k_2)^2$, where $p_n^2 = m_B^2$ and $k_2^2 = m_\pi^2$, which gives for k_1^2 the allowed range $k_1^2 \leq (m_B - m_\pi)^2$. We notice that, unlike the other branch points discussed above, which are determined by the lowest intermediate states in the unitarity sum, the range of the variable k_1^2 has a more kinematical nature.

We express now the whole amplitude in terms of its discontinuity by a dispersion integral. Assuming that one subtraction is necessary and including the degenerate terms in the subtraction constant, we obtain

$$\begin{aligned} A(B^0 \to \pi^+\pi^-) &= A(\kappa_0^2) + \frac{(m_\pi^2 - \kappa_0^2)}{\pi} \\ &\times \int_{-\infty}^{(m_B-m_\pi)^2} \mathrm{d}k_1'^2 \frac{\sigma_{\mathrm{FSI}}(k_1'^2)}{(k_1'^2 - m_\pi^2 - \mathrm{i}\epsilon)(k_1'^2 - \kappa_0^2)} \\ &+ \frac{(m_\pi^2 - \kappa_0^2)}{\pi} \int_{9m_\pi^2}^{\infty} \mathrm{d}k_1'^2 \frac{\sigma_1(k_1'^2)}{(k_1'^2 - m_\pi^2)(k_1'^2 - \kappa_0^2)} \\ &+ \frac{(m_\pi^2 - \kappa_0^2)}{\pi} \int_{(m_B+m_\pi)^2}^{\infty} \mathrm{d}k_1'^2 \frac{\sigma_2(k_1'^2)}{(k_1'^2 - m_\pi^2)(k_1'^2 - \kappa_0^2)}, \end{aligned} \quad (11)$$

where σ_{FSI} is given formally by the same expression (10) given above, evaluated now at fixed $s = m_B^2$ and variable k_1^2, and

$$\begin{aligned} \sigma_1 &\approx \sum_{n=3\pi,\dots} \delta(k_1 + p_n) A^*(n \to \pi B) M(\pi \to n), \\ \sigma_2 &\approx \sum_{n=B\pi,\dots} \delta(k_1 + p_n) M^*(n \to \pi B) A(\pi \to n). \end{aligned} \quad (12)$$

We denoted generically, as before, by $A(M)$ the weak (strong) amplitudes, respectively.

It is easy to see that both dispersion relations (9) and (11) lead to the same discontinuity on shell, equal to the spectral function σ_{FSI} evaluated for physical masses. These relations might be useful in principle if the decay amplitude can be calculated (by chiral theory, lattice, etc.) at some particular points ($s = s_0$ or $k_1^2 = \kappa_0^2$), with a better accuracy than at the physical points, $s = m_B^2$ and $k_1^2 = m_\pi^2$, respectively. The complete evaluation of the dispersion relations is very difficult, since they involve off-shell quantities in the spectral functions. However, as we will show below, the dispersion relation (11) can be written in a different form, more convenient for the study of the final state interactions. We first remark that the spectral function σ_{FSI} defined in (10) is actually independent of the dispersive variable k_1^2, for fixed $(k_1 + k_2)^2 = m_B^2$. This can be easily seen by performing the phase space integral in the first sum of (8) in the c.m. system, when $\boldsymbol{p}_n = \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0$ and the energy squared is equal to m_B^2. As the weak decay amplitude (the matrix element $\langle n|\eta_B|0\rangle/(2p_0)^{1/2}$ in (8)) and the invariant strong amplitude (the matrix element $\langle \pi^-(k_2)|\eta_{\pi^+}(0)|n\rangle/(2k_{01})^{1/2}$) depend both only on the Mandelstam variables and the physical masses of the particles involved, the result of the phase space integration at fixed s is independent of k_1^2 and contains only on-shell quantities[1].

We notice that the most general form of a function having a branch cut for $-\mu^2 \le k_1^2 \le (m_B - m_\pi)^2$, with a constant discontinuity σ_{FSI}, is

$$A_{\mathrm{FSI}}(k_1^2) = \mathcal{P}(k_1^2) + \frac{\sigma_{\mathrm{FSI}}}{\pi} \ln\left[\frac{k_1^2 - (m_B - m_\pi)^2}{\mu^2 + k_1^2}\right], \quad (13)$$

where $\mathcal{P}(k_1^2)$ is a polynomial (more generally, an entire function), independent on σ_{FSI}. In order to construct the full decay amplitude we must add to the function A_{FSI} the contribution of the degenerate terms and the last two dispersion integrals in (11), with possible subtractions. All these terms, as well as the polynomial $\mathcal{P}(k_1^2)$, are independent of the discontinuity σ_{FSI}. Therefore, by combining them into a single constant and choosing the scale $\mu^2 \approx m_B^2$, we write the physical amplitude as

$$\begin{aligned} A(B_d^0 \to \pi^+\pi^-) &= A_0 + \frac{\sigma_{\mathrm{FSI}}}{\pi} \ln\left[\frac{m_\pi^2}{(m_B - m_\pi)^2} - 1\right], \end{aligned} \quad (14)$$

where A_0 is the genuine contribution which remains when the long distance final state interactions are switched off, i.e. $\sigma_{\mathrm{FSI}} = 0$. It is worth mentioning that this separation of the final state interactions from the other parts of the dynamics was possible only due to the fact that σ_{FSI} does not depend on the dispersive variable, allowing us to construct A_{FSI} according to (13). The subtraction constants in the usual subtracted dispersion relations, like (9) or (11), do not have a similar interpretation: they represent the values of the amplitude at some particular points, and depend implicitly on all the spectral function in the dispersion relation.

We shall end this section with some comments. First, we recall that rigorous analytic properties in the external momenta are proved in axiomatic field theory only for a small region close to the physical masses [33]. Therefore, the dispersion relations presented above can be accepted only as an heuristic conjecture, whose validity remains to be tested. We mention also that the off-shell analytic continuation in external momenta is in general plagued by ambiguities. They may appear, in the present formalism, in the evaluation of the degenerate terms and of the off-shell amplitudes entering the spectral functions. Moreover, we notice that even the analytic properties of the off-shell amplitude may depend on the specific expression of the on-shell amplitude, used as starting point of the extrapolation. For instance, by applying the LSZ procedure to the expression (1) of the amplitude, we obtained in [25] only a part of the dispersive branch cuts written in (9) and (11). The contribution of the missing dispersion integrals (namely, the FSI contribution in (9) and the ISI contribution in (11)) is hidden in the corresponding degenerate terms, which have a different form [25]. Of course, one expects that the amplitude on-shell is recovered in an univocal way, but the compensation of the ambiguities of various terms is difficult to see in approximate calculations.

As concerns the phenomenological applications, the final state interactions in B hadronic decays were investigated the last years by means of dispersion relations with respect to the momentum squared (s) of the B meson [18–21] (recently, this method was applied also to $K \to \pi\pi$ decay [32–36]). These dispersion relations look more familiar, due to their formal resemblance with the case of the

[1] The unusual property of the spectral function to be independent of the dispersive variable k_1^2, was noticed a long time ago [30]

pion form factor. However, as seen from (9), the similarity is not complete, due to the presence of the initial strong interactions in the weak decay and the appearance of off-shell quantities. The dispersion relations with respect to the momentum squared k_1^2, which seem less intuitive, were applied to B decays in [25] (we mention also one earlier application of this technique for the calculation of the nucleon form factor [31]). In Sect. 4 we shall use this type of dispersion relations, written in the convenient form (14), for discussing the effects of the final state interactions in the $B^0 \to \pi^+\pi^-$ decay. Before making this analysis, we shall first investigate the spectral function $\sigma_{\rm FSI}$ appearing in this relation.

3 The Goldberger–Treiman procedure

Following [9], we shall use the parametrization

$$\begin{aligned} &A(B_d^0 \to \pi^+\pi^-) \\ &= \mathrm{i}\frac{G_{\rm F}}{\sqrt{2}} m_B^2 f_+(m_\pi^2) f_\pi |V_{ud}V_{ub}^*| \mathrm{e}^{\mathrm{i}\gamma} \\ &\quad \times \left[T_u(B \to \pi^+\pi^-) + \frac{\mathrm{e}^{-\mathrm{i}\gamma}}{R_b} T_c(B \to \pi^+\pi^-) \right], \quad (15) \end{aligned}$$

obtained by extracting from the amplitudes A_u and A_c of (3) the "naive" factorized amplitude, expressed in terms of the pion decay constant f_π and the $B \to \pi$ transition form factor $f_+(m_\pi^2)$ ($R_b = |V_{ub}/(\lambda V_{cb})|(1-\lambda^2/2) \approx 0.377$). The spectral function $\sigma_{\rm FSI}$ defined in (10) can be written in a similar way as

$$\begin{aligned} \sigma_{\rm FSI} = \mathrm{i}\frac{G_{\rm F}}{\sqrt{2}} m_B^2 f_+(m_\pi^2) f_\pi |V_{ud}V_{ub}^*| \mathrm{e}^{\mathrm{i}\gamma} \\ \times \left[\sigma_{\rm FSI}^u + \frac{\mathrm{e}^{-\mathrm{i}\gamma}}{R_b} \sigma_{\rm FSI}^c \right], \quad (16) \end{aligned}$$

where, recalling that $\sigma_{\rm FSI}$ is given by the first term in the unitarity sum (8), we have

$$\begin{aligned} &\sigma_{\rm FSI}^j \sim \sum_n \delta(k_1 + k_2 - p_n) \langle \pi^- | \eta_{\pi^+} | n \rangle \langle n | \eta_B^j | 0 \rangle, \\ &j = u, c. \quad (17) \end{aligned}$$

We assumed here that the source η_B admits a decomposition in two terms, analogous to that of the weak Hamiltonian (2). This shows also that one can derive separate dispersion relations for each of the amplitudes T_u and T_c. In particular, according to (14), we shall write these relations in the form

$$T_j = T_{j,0} + \frac{\sigma_{\rm FSI}^j}{\pi} \ln \left[\frac{m_\pi^2}{(m_B - m_\pi)^2} - 1 \right], \quad j = u, c, \quad (18)$$

where we denoted by $T_{j,0}$ the analog of the term A_0 in the relation (14), divided by the constant factorized in (15).

For further applications, it is important to notice that the spectral functions $\sigma_{\rm FSI}^u$ and $\sigma_{\rm FSI}^c$ are real quantities:

$$\sigma_{\rm FSI}^j = (\sigma_{\rm FSI}^j)^*, \quad j = u, c. \quad (19)$$

The proof of these equalities is based on the properties of the matrix elements in (17) under the PT transformation [30]. More precisely, in the present case, we have [25]

$$\begin{aligned} \langle \pi^-(k_2) | \eta_{\pi^+}(0) | n, \mathrm{in} \rangle &= \langle \pi^-(k_2) | \eta_{\pi^+}(0) | n, \mathrm{out} \rangle^*, \\ \langle n, \mathrm{in} | \eta_B^j(0) | 0 \rangle &= -\langle n, \mathrm{out} | \eta_B^j(0) | 0 \rangle^*, \quad (20) \end{aligned}$$

where the minus sign in the second relation is due to the specific spin–parity properties of the relevant part of the weak hamiltonian. Using the equalities (20) in (17), and taking into account the equivalence of the complete sets of "in" and "out" intermediate states in the unitarity sum, it is easy to prove the relations (19) (the minus sign in (20) is compensated by leaving aside an imaginary constant in the definition (16)).

In approximate calculations, the set of intermediate states in the unitarity sum (17) is truncated, which might lead to violations of the reality conditions (19), and to the appearance of artificial strong phases in the spectral functions. This fact is important in the present case, since the unitarity sum is evaluated at large c.m. energy squared, $s = m_B^2$, where many inelastic channels are open. However, with a "good" choice of the truncated set one can avoid the appearance of unphysical phases. The idea of Goldberger and Treiman [37] was to take the intermediate states in the symmetric combination $1/2|n,\mathrm{in}\rangle\langle n,\mathrm{in}| + 1/2|n,\mathrm{out}\rangle\langle n,\mathrm{out}|$, which represents also a complete set. The remarkable point is that, even when it is truncated, this set generates spectral functions which satisfy the reality condition (19), at each step of the approximation. The symmetric summation simulates therefore, in a certain measure, the effects of inelastic states, without incorporating them explicitly in the unitarity sum[2].

It is worth mentioning that a complete set written in a symmetric form is quite natural in the LSZ method: indeed, when deriving the discontinuity of the amplitude, the initial $\theta(x_0)$ function in (7), whose origin is the reduction of an "out" pion, is actually replaced by $\theta(x_0)/2 + \theta(-x_0)/2$ [30]. This means that the final two-pion state appears in the discontinuity in the symmetric combination $1/2|\pi^+\pi^-,\mathrm{out}\rangle + 1/2|\pi^+\pi^-,\mathrm{in}\rangle$ [3]. It is therefore reasonable to take the same symmetric combination also for the intermediate states n.

The Goldberger–Treiman procedure allows us to write the spectral functions $\sigma_{\rm FSI}^j$ defined in (17) as

$$\begin{aligned} \sigma_{\rm FSI}^j = \frac{1}{2} \sum_n \delta(k_1 + k_2 - p_n) [M^*(n \to \pi^+\pi^-) T_j(B \to n) \\ + M(n \to \pi^+\pi^-) T_j^*(B \to n)], \quad j = u, c, \quad (21) \end{aligned}$$

where $M(n \to \pi^+\pi^-)$ denotes the amplitude of the strong transition from the intermediate state n to the final $\pi^+\pi^-$ state, and $T_j(B \to n)$ is the specific part of the weak decay

[2] An alternative approach to include the effects of the inelastic states in B hadronic decays, based on statistical arguments and Regge phenomenology, was proposed in [26,27]

[3] A similar symmetric combination of "in" and "out" states was obtained in a related context in [38]

amplitude of B into the same intermediate state (divided by the constant factorized in (15) and (16)).

It is known that the strong dynamics at high energies is dominated by multiparticle production. However, as argued in [27], the contribution of the multiparticle intermediate states in B decay is suppressed by a flavour mismatch between the weak and the strong parts of the process. Therefore, only the states composed of two low mass resonances are expected to yield an important contribution to the rescattering. As shown in [27], this picture is consistent with the absence of final state interactions in B decays in the heavy mass limit [9], since the production of two resonances is expected to vanish at high energies.

In the two-particle approximation of the unitarity sum, the weak decay amplitudes are completely specified by the masses of the particles in the intermediate states, being independent on the phase space variables [25]. Then the phase space integration implicit in (21) can be performed exactly, leading to

$$\begin{aligned}\sigma^j_{\mathrm{FSI}} = \frac{1}{2}\sum_{\bar{P}_aP_a,\lambda} & [M^*_{0,\lambda}(\bar{P}_aP_a \to \pi^+\pi^-)T^\lambda_j(B \to \bar{P}_aP_a) \\ & + M_{0,\lambda}(\bar{P}_aP_a \to \pi^+\pi^-)T^{\lambda*}_j(B \to \bar{P}_aP_a)] \\ & + \ldots ,\end{aligned} \tag{22}$$

where $M_{0,\lambda}(\bar{P}_aP_a \to \pi^+\pi^-)$ denote the S-wave projection of the strong amplitudes and a summation over the helicities λ of the intermediate states must be peformed in general. By inserting this expression in (18), and writing explicitly the real and the imaginary part of the logarithm, we obtain the relation

$$\begin{aligned} & T_j(B \to \pi^+\pi^-) \\ & = T_{j,0}(B \to \pi^+\pi^-) + \left[\sum_{\bar{P}_aP_a,\lambda} \mathrm{Re}[M^*_{0,\lambda}(\bar{P}_aP_a \to \pi^+\pi^-)\right. \\ & \quad \left. \times T^\lambda_j(B \to \bar{P}_aP_a)] + \ldots \right] \\ & \quad \times \left[\mathrm{i} + \frac{1}{\pi}\ln\left(1 - \frac{m_\pi^2}{(m_B - m_\pi)^2}\right)\right], \quad j = u, c. \end{aligned} \tag{23}$$

The sum includes the dominant two-particle quasielastic states and resonances $\bar{P}_aP_a$, and the dots represent the contribution of the multiparticle states.

The strong amplitudes entering (23) are evaluated at the c.m. energy squared equal to $m_B^2 \approx 25\,\mathrm{GeV}^2$. For low masses of the intermediate particles we can use the generic Regge amplitude [39]

$$-\gamma(t)\frac{\tau + \mathrm{e}^{-\mathrm{i}\pi\alpha(t)}}{\sin\pi\alpha(t)}\left(\frac{s}{s_o}\right)^{\alpha(t)}, \tag{24}$$

where $\gamma(t)$ is the residue function, τ the signature, $\alpha(t) = \alpha_0 + \alpha' t$ the linear trajectory, and $s_0 \approx 1\,\mathrm{GeV}^2$. Modifications of the standard Regge expression when the particles have larger masses are discussed in [27]. The S-wave projection of the amplitude (24) in the spinless case is

$$\begin{aligned} & M_{0,0}(\bar{P}_aP_a \to \pi^+\pi^-) \\ & \approx \xi\frac{\gamma(0)}{32\pi\alpha'}\frac{1}{m_BqL}\mathrm{e}^{(\alpha_0+\alpha' t_0+2\alpha' qq')L} \\ & \quad \times \left[1 - \mathrm{e}^{-2\alpha' qq' L}\right], \end{aligned} \tag{25}$$

where $q = 1/2(m_B^2 - 4m_\pi^2)^{1/2}$ and $q' = 1/2(m_B^2 - 4m_a^2)^{1/2}$ are the c.m.s. momenta of the final and intermediate state, respectively, $t_0 = -(q^2 + q'^2)$, $L = \ln m_B^2/s_0 - \mathrm{i}\pi/2$ and ξ is a signature factor (equal, in particular, to -1 for the pomeron and $\mathrm{i}2^{1/2}$ for the ρ trajectory [25]). In deriving (25) we neglected the t dependence of the ratio $\gamma(t)/\sin(\pi\alpha(t)/2)$ for $\tau = 1$ trajectories, and of the ratio $\gamma(t)/\cos(\pi\alpha(t)/2)$ for $\tau = -1$.

4 Quark–hadron duality

The relation between the QCD predictions and the hadronic physics is an extremely complex, still unsolved problem. For testing quark–hadron duality it is in principle necessary to perform an analytic continuation from the spacelike region of momenta, where operator product expansions and perturbative QCD are valid, to the timelike axis, where the physical processes are described in terms of hadronic degrees of freedom. This procedure, based on dispersion relations, was applied to simple objects, like the current–current vacuum correlation functions or the electromagnetic form factors.

The weak hadronic decays are much more complicated, due to the presence of hadrons in both initial and final states. Usually, the strong processes at scales larger than m_b are integrated out, being included in the Wilson coefficients entering the effective hamiltonian (2). In perturbative QCD, the decay amplitudes are treated in the heavy mass limit using the framework of perturbative factorization for exclusive processes, based on hard scattering kernels and light-cone distribution amplitudes, with the heavy mass playing the same role as the large momentum transfer.

In the hadronic picture, we shall consider the dispersion relation (18) (written in more detail in (23)) where, as discussed in Sect. 3, the first term $T_{j,0}$ is given by the contributions which remain after switching off the final state strong interactions among the emitted pions. This term should be provided, in principle, by a nonperturbative calculation, which excludes in a systematic way the final state interactions. As such a calculation is lacking, we resort to a qualitative discussion, based on quark–hadron duality.

We recall that, according to the discussion below (14), the terms $T_{j,0}$ include the degenerate terms and the last two dispersive integrals in the relation (11). In all these terms, the two final pions appear in different matrix elements, and may be associated qualitatively to diagrams with no gluon exchanges between them. In particular, in the spirit of the dispersive formalism, the last two integrals in (11) may be interpreted as “initial state interactions”, in a crossed channel. Therefore, it is reasonable to assume that a considerable part of $T_{j,0}$ consists of the naive factorized amplitude, which is associated to processes with no gluon exchanges between the emitted pion π^+ and the system $(\pi^- B_d^0)$ (except those already included in the Wilson coefficients and the short distance processes taking place before hadronization). As for the last term in the relation (14) (or (23)), which describes in the hadronic picture the final state strong interactions, it is dual to the topologies (penguin annihilation, exchange diagrams, scattering of the spectator quark, vertex corrections to the emission diagrams), involving gluon exchanges between the final pions. Of course, the correspondence between various quark diagrams and the terms appearing in the hadronic formalism is not simple beyond the lowest orders of perturbation theory. In particular, it is impossible to associate in an univocal way the diagrams involving many gluons either to the initial, or the final state interactions. Moreover, from the point of view of final state interactions, the spectator quark does not play a special role. Therefore, the vertex corrections involving the quarks emitted in the weak process, which are included in the factorized part of the amplitude in the standard QCD factorization approach [9], contribute also to the final state interaction part of the amplitude.

In the QCD factorization approach [9], the dominant contribution to the decay amplitude is given by the factorized term, with corrections which are suppressed, in the heavy limit $m_b \to \infty$, either by powers of $\alpha_s(m_b)$, or by powers of $\Lambda_{\rm QCD}/m_b$. As discussed recently, the power suppressed corrections might be enhanced by pure soft effects, such as end point singularities and higher twist terms in the pion distribution amplitudes, appearing in annihilation diagrams or in the hard spectator interactions. A recent evaluation of these corrections in perturbative QCD factorization was given in [14], but there are still differences between the results obtained by different authors.

In what follows we shall estimate the heavy mass corrections produced by the final state interactions, using the hadronic dispersion relation written in the form (23). We shall consider first the contribution of the elastic and quasielastic rescattering, taking as intermediate states $\bar{P}_a P_a$ the lowest pseudoscalar mesons $\pi^+\pi^-$, $\pi^0\pi^0$, $\bar{K}^0 K^0$, K^+K^- and $\eta\eta$. The effect of higher resonances describing inelastic rescattering will be discussed below.

We assume that the amplitudes T_j appearing in (23) can be expanded in the heavy quark limit, i.e. for large $m_B \approx m_b$, as

$$T_j \approx T_{j,0} + O(\alpha_s(m_B)) + O(\Lambda/m_B) + \ldots , \qquad (26)$$

where $T_{j,0}$ are approximately given by the factorized amplitudes, with short distance corrections. These values have small imaginary parts, produced by the complex effective Wilson coefficients, vertex corrections or short distance effects in the penguin and annihilation diagrams.

It is easy to write down the high energy limit of all the quantities entering (23). First, from the explicit expression of the logarithm in the r.h.s. of this relation, it follows that the real part of FSI amplitude is suppressed by two powers of the heavy mass with respect to the imaginary part. The heavy limit behavior of the Regge amplitude (25) depends of the specific trajectory. For the pomeron (with $\alpha_0 \approx 1.0$ and $\alpha' \approx 0.25$) we obtain the expansion

$$M_0^{(\mathrm{P})} \approx \frac{\gamma_{\mathrm{P}}(0)}{4\pi} \left[\frac{\mathrm{i}}{\ln \frac{m_B^2}{s_0}} - \frac{\pi}{2} \frac{1}{\ln^2 \frac{m_B^2}{s_0}} + \ldots \right] , \qquad (27)$$

which shows that the dominant contribution in the heavy mass limit is imaginary. At the physical scale, using $\gamma_{\mathrm{P}}(0) \approx 25.6$ [25], we obtain $M_0^{(\mathrm{P})} \approx -0.23 + 0.69\mathrm{i}$.

For a physical trajectory, like ρ, using $\alpha_0 \approx 0.5$ and $\alpha' \approx 1$, we obtain

$$M_0^{(\rho)} \approx \frac{\gamma_\rho(0)}{16\pi m_B} \left[\frac{\mathrm{i}+1}{\ln \frac{m_B^2}{s_0}} - \frac{\pi}{2} \frac{1-\mathrm{i}}{\ln^2 \frac{m_B^2}{s_0}} + \ldots \right] . \qquad (28)$$

This amplitude is suppressed by one power of m_B compared to the pomeron amplitude (27). At the physical scale, using $\gamma_\rho(0) \approx 31.4$ [25], we obtain for the ρ trajectory $M_0^\rho \approx 0.015 + 0.047\mathrm{i}$. The masses of the particles undergoing the strong scattering appear as power suppressed terms in the expansions (27) and (28).

By inserting the expansions (26)–(28) in (23), we can derive iteratively the magnitude of the coefficients of the logarithmic and power corrections in the heavy limit expansion (26). As an illustration of the method, we take as input of the iterative procedure the values

$$T_0 = T_{u,0} = 0.969 - 0.017\mathrm{i}, \qquad P_0 = \frac{T_{c,0}}{R_b} = 0.246 + 0.03\mathrm{i}, \qquad (29)$$

which are typical for the factorized amplitude with short distance corrections [14].

Using these values as the lowest approximation of the amplitudes T_j in the right hand side of the relation (23), we obtain to first order

$$T = T_{u,1} \approx 0.969 - 0.23\mathrm{i}, \qquad P = \frac{T_{c,1}}{R_b} \approx 0.246 + 0.00012\mathrm{i}. \qquad (30)$$

The dominant corrections are given by the pomeron contribution to the elastic channel. In particular, the imaginary part of $T_{u,1}$ is due mainly to the next to leading order (second) term in the expansion (27) of the pomeron amplitude, since the dominant term is suppressed by the symmetric Goldberger–Treiman summation in (23).

The values (30) have uncertainties due to the higher order terms in the heavy mass expansion. First we notice that the contribution of the pseudoscalar mesons $\pi^0\pi^0$, $\bar{K}^0 K^0$, K^+K^- and $\eta\eta$, responsible for the quasielastic rescattering in the unitarity sum of (23), is negligible. Indeed, these states are produced by non-dominant decay diagrams, and the corresponding Regge amplitudes are described by physical trajectories, which are suppressed with respect to the pomeron. The estimates made in [25], based on $SU(3)$ flavor symmetry, show that the effect of these channels on the spectral functions is not larger than several percents.

The inelastic states might give an important contribution in the unitarity sum if they are produced by dominant decay diagrams, or the corresponding CKM coefficients are large. The first states which can contribute are the lowest vector resonances, $\rho^+\rho^-$, $\rho^0\rho^0$, $\bar{K}^*K^*$, $\omega\omega$ and $\phi\phi$, which describe intermediate states with four or six pions. The quark diagrams for B decays into VV pairs are similar to those of the corresponding pseudoscalar mesons. Therefore, only the pair $\rho^+\rho^-$ is produced by a dominant tree diagram. Estimates based on factorization [40] give for the helicity amplitudes of the weak decay $B_d^0 \to \rho^+\rho^-$ values comparable, up to a factor of 2, with the amplitude of the decay $B_d^0 \to \pi^+\pi^-$. As concerns the strong amplitude $M(\rho^+\rho^- \to \pi^+\pi^-)$, it is described in the Regge model by the exchange of ω and A_2, with trajectories $\alpha \approx 0.5 + t$ (almost degenerate with that of ρ), and the π exchange, whose trajectory $\alpha_\pi \approx 0. + 0.9t$ is non-dominant [39]. As discussed in [41], the kinematic factors of the t-channel helicity amplitudes suppress in the present case the contribution of the natural parity exchanges ω and A_2, so that the scattering amplitude of $\rho^+\rho^- \to \pi^+\pi^-$ is described at small t by π exchange. This implies a considerable suppression, verified indirectly in the related reactions of vector meson production $\pi^\pm N \to \rho^\pm N$ [42,43]. Therefore, the vector meson resonances seem to yield a negligible contribution to the inelastic rescattering in $B_d^0 \to \pi^+\pi^-$ decay. Using this spin suppression argument, we expect that the same conclusion applies also to other vector mesons of higher mass.

The role of the intermediate states with charm, like the pair $\bar{D}D$, has been considered by several authors and is still controversial. Since a large part of the inelastic $\pi\pi$ scattering at $s^{1/2} \approx m_B$ goes into multiparticle states composed of noncharmed mesons, the contribution of these states was assumed in [27] to be negligible. The annihilation of the $\bar{c}c$ pair in the penguin diagrams proceeds therefore through a short distance interaction, which can be computed perturbatively. We mention however that the possibility of a long distance contribution of the "charming penguins" was also considered by some authors [44], possibly through the intermediate state $\bar{D}D$ [23,24]. A detailed estimate is difficult since the validity of the Regge model is questionable at $s^{1/2} \approx m_B$ for large masses ($m_D = 1.68\,\mathrm{GeV}$) of the particles undergoing the strong rescattering [27]. We mention that the contribution of high mass resonances, such as $\bar{D}D$, is actually suppressed by the phase space integral in (21), evaluated at fixed $s = m_B^2$. In the present approach, we assume also that their global effect is taken into account in a certain measure by the Goldberger–Treiman symmetric summation. With this assumption, we do not expect other important power corrections in the spectral functions.

From the values given in (30) we obtain

$$\left|\frac{P}{T}\right| \approx 0.246, \quad \delta = \mathrm{Arg}\left[\frac{P}{T}\right] \approx 14^\circ, \tag{31}$$

while the input values (29) correspond to $|P_0/T_0| = 0.25$ and $\mathrm{Arg}[P_0/T_0] = 8^\circ$. Thus, in the present approach the final state interactions do not considerably modify the modulus and the phase of the ratio P/T. We recall that the lowest order values $T_{j,0}$ depend on the renormalization scale μ in the Wilson coefficients. It is expected that this dependence will be diminished by the inclusion of other terms in the l.h.s. of (23).

5 Conclusions

In the present work we investigated the effects of the final state interactions to $B_d^0 \to \pi^+\pi^-$ decay in a formalism based on hadronic unitarity and dispersion relations. We discussed the heuristic derivation of the dispersion relations with respect to the momenta of the external particles, by applying the LSZ procedure to the S-matrix element of the weak decay. The ambiguities which affect in general the off-shell extrapolation of the amplitudes appear in our formalism in the so-called "degenerate terms", produced by the equal-time commutators in the LSZ formalism, and in the off-shell quantities entering the spectral functions.

The evaluation of the dispersion relations in external momenta is in general very complicated. However, we noticed that the dispersion relation with respect to the momentum squared of one final pion can be written in the convenient form (14), where the contribution of the final state interactions is separated from other terms, and involves only on-shell quantities. We used this relation, written in more detail in (23), as an iterative scheme for determining the corrections to the factorized amplitude, generated by the final state interactions in the heavy mass limit. A nontrivial prediction of the formalism is that the real part of the FSI contribution is suppressed by two powers of the heavy mass, compared to the imaginary part. Using for illustration a numerical input value suggested by QCD factorization, we noticed the dominant effect of the next to leading logarithmic term of the pomeron contribution. Other sources of large power corrections to the factorized amplitude are not found. We discussed the contribution of the lowest pseudoscalar mesons and vector meson resonances, and assumed that the effects of higher resonances and multiparticle states are qualitatively taken into account by the Goldberger–Treiman method of calculating the spectral functions. In particular, the results of a numerical test indicate that the phase and the modulus of the ratio P/T are not drastically modified by the final state interactions. Using improved results of QCD calculations, it will be possible to test the dispersion relations conjectured in the present work and, more generally, the validity of quark–hadron duality.

Acknowledgements. This work was realized in the frame of the Cooperation Agreements between IN2P3 and NIPNE-Bucharest, and between the CNRS and the Romanian Academy. Two of the authors (I.C. and L.M.) received financial support from the Romanian National Agency for Science, Technology and Innovation, under the Grant Nr. 6126/2000.

References

1. P. Ball et al., hep-ph/0003238, published in CERN Yellow Report CERN-2000-004, 1999 CERN Workshop on Standard Model Physics (and more) at the LHC, edited by G. Altarelli, M. Mangano
2. D. Cronin-Hennessy et al., (CLEO Collaboration), Phys. Rev. Lett. **85**, 515 (2000)
3. G. Cavoto (BaBar Collaboration), talk at the 36th Rencontres de Moriond on QCD and Hadronic Interactions, Les Arcs, France, 17–24 March 2001, to appear in the proceedings, hep-ex/0105018
4. T. Iijima (Belle Collaboration), talk at the 4th International Conference on B Physics and CP Violation (BCP 4), Ago Town, Japan, 19–23 February 2001, to appear in the proceedings, hep-ex/0105005
5. M. Bauer, B. Stech, M. Wirbel, Z. Phys. C **34**, 103 (1987); Z. Phys. C **29**, 637 (1985)
6. M.J. Dugan, B. Grinstein, Phys. Lett. B **255**, 583 (1991)
7. J.F. Donoghue, A.A. Petrov, Phys. Lett. B **393**, 149 (1997)
8. A. Ali, C. Greub, Phys. Rev. D **57**, 2996 (1998)
9. M. Beneke, G. Buchalla, M. Neubert, C.T. Sachrajda, Phys. Rev. Lett. **83**, 1914 (1999); Nucl. Phys. B **591**, 313 (2000)
10. M. Beneke, G. Buchalla, M. Neubert, C.T. Sachrajda, Contribution to ICHEP2000, 27 July–2 August, Osaka, Japan, PITHA-00-13, hep-ph/0007256
11. M. Beneke, PITHA 00-23, hep-ph/0009328
12. D. Du, D. Yang, G. Zhu, Phys. Lett.B. **488**, 46 (2000)
13. T. Muta, A. Sugamoto, M.Z Yang, Y.D. Yang, Phys. Rev. D **62**, 094020 (2000)
14. M. Beneke, G. Buchalla, M. Neubert, C.T. Sachrajda, hep-ph/0104110
15. Y.Y. Keum, H. Li, A. Sanda, Phys. Rev. D **63**, 054008 (2001);
    Y.Y. Keum, H. Li, Phys. Rev. D **63**, 074006 (2001)
16. C.D. Lu, K. Ukai, M.Z. Yang, Phys. Rev. D **63**, 074009 (2001)
17. J.F. Donoghue et al., Phys. Rev. Lett. **77**, 2178 (1996)
18. B. Blok, I. Halperin, Phys. Lett. B **385**, 324 (1996)
19. J.F. Donoghue, E. Golowich, A.A. Petrov, Phys. Rev. D **55**, 2657 (1997)
20. B. Blok, M. Gronau, J.L. Rosner, Phys. Rev. Lett. **77**, 3999 (1997)
21. A.F. Falk, A.L. Kagan, Y. Nir, A.A. Petrov, Phys. Rev. D **57**, 4290 (1998)
22. D. Delépine, J.-M. Gérard, J. Pestieau, J. Weyers, Phys. Lett. B **429**, 106 (1998);
    J.-M. Gérard, J. Pestieau, J. Weyers, Phys. Lett. B **436**, 363 (1998)
23. A.N. Kamal, Phys. Rev. D **60**, 094018 (1999)
24. Z. Xing, Phys. Lett. B **493**, 301 (2000)
25. I. Caprini, L. Micu, C. Bourrely, Phys. Rev. D **60**, 074016 (1999); D **62**, 034016 (2000)
26. M. Suzuki, L. Wolfenstein, Phys. Rev. D **60**, 074019 (1999)
27. P. Zenczykowski, Phys. Rev. D **63**, 014016 (2001); hep-ph/0102186
28. H. Lehmann, K. Symanzik, W. Zimmermann, Nuovo Cimento, **1**, 205 (1956); **2**, 425 (1957)
29. M.L. Goldberger, K.M. Watson, Collision theory (Wiley, New York 1964)
30. G. Barton, Introduction to dispersion techniques in field theory (Benjamin, New York 1965)
31. A.D. Bincer, Phys. Rev. **118**, 855 (1960)
32. M. Suzuki, hep-ph/0102028
33. G. Kälen, A.S. Wightman, K. Dan. Vidensk. Fys. Skr. **1**, (1958) No. 6
34. E. Pallante, A. Pich, Phys. Rev. Lett. **84**, 2568 (2000)
35. A.J. Buras et al., Nucl. Phys. B **565**, 3 (2000)
36. M. Bücher, G. Colangelo, J. Kambor, F. Orellana, hep-ph/0102387, hep-ph/0102389
37. M.L. Goldberger, S.B. Treiman, Phys. Rev. **110**, 1178 (1958); **111**, 354 (1958)
38. L. Maiani, M. Testa, Phys. Lett. B **245**, 585 (1990)
39. P.D.B. Collins, Introduction to Regge theory and high energy physics (Cambridge University Press, Cambridge, England 1977)
40. A. Ali, G. Kramer, C.D. Lü, Phys. Rev. D **58**, 094009 (1998)
41. L. Jones, Phys. Rev. **163**, 1523 (1967)
42. J.C. Pratt et al., Phys. Lett. B **41**, 383 (1972)
43. G.F. Fox, C. Quigg, Ann. Rev. Nucl. Sci (1973) 219
44. M. Ciuchini, E. Franco, G. Martinelli, L. Silvestrini, Nucl. Phys. B **501**, 271 (1997);
    M. Ciuchini et al., hep- ph/0104126